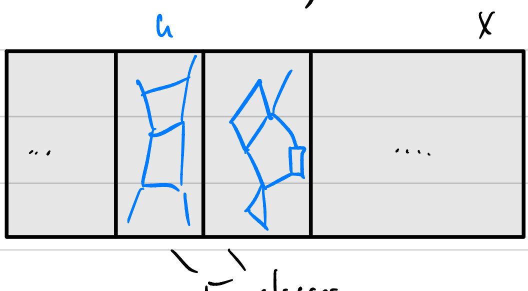
Ergodic Theory and Measured Group Theory
Lecture 24

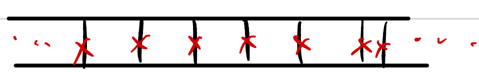
Recall the following thought: treeings are minimal graphings, must they achieve the cost of the equivalence relation?

Prop. Let E be a pmp CBER on (X,μ). Let G be a (Borel) graphing of E that achieves the cost, i.e. $c_\mu(E) = C_\mu(G)$. Then G is a treeing a.e.

Proof.



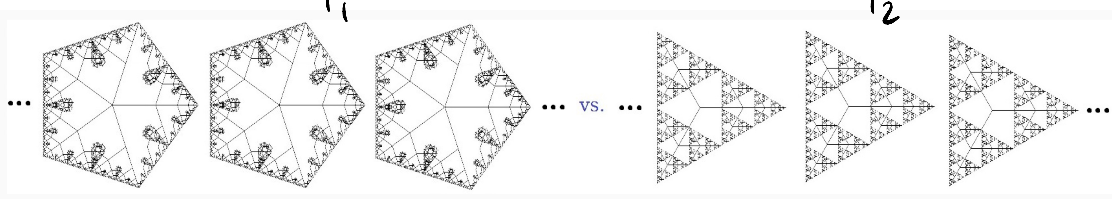
We need to delete a positive measure of edge at once and we can't just say something like "delete the least edge from each cycle" because although the result would indeed be a Borel forest (Minimal Subforest), it won't be a graphing of E:



We need to do so that the resulting graph has the same components as G. Using Feldman–Moore, one can show that a Borel version of Zorn's lemma holds for CBERs, yielding a maximal collection $\mathcal{C}$ of disjoint cycles that is Borel as a subset of $X^{<\mathbb{N}} := \bigcup_{n \in \mathbb{N}} X^n$. By maximality, it should intersect every G-component that has a cycle, which is a positively-measured set by our contradictory assumption. Thus, $\bigcup \mathcal{C}$ is also positive measure, hence removing one edge (say least) from each cycle in $\mathcal{C}$ reduces the cost of G, a contradiction. $\square$

But again does every treeing achieve the cost? Can't we have two treeings of the same E one bushier than another?

T_1 T_2



E E

$C_\mu(T_1) = \frac{5}{2}$ and $C_\mu(T_2) = \frac{3}{2}$?

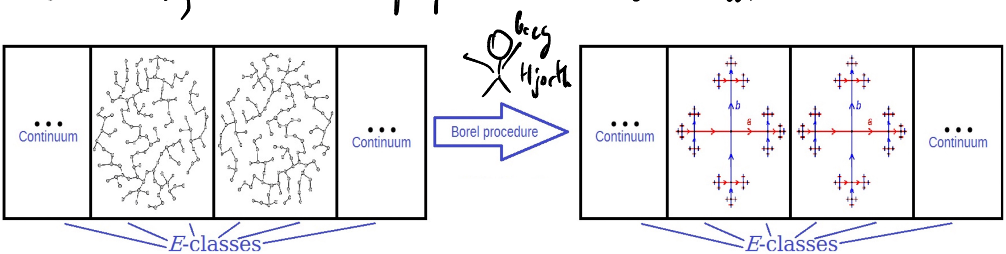
Fundament theorem of cost (Gaboriau 1997). Any treeing T of pmp CBER E achieves the cost of E, i.e. $c_\mu(E) = C_\mu(T)$. In particular, any two treeings have equal cost.

Corollary (Gaboriau). For each $n \le \infty$, any free pmp action of $\mathbb{F}_n$ induces an orbit eq. rel. of cost $= n$.

In particular, for $m \neq n$, the orb. eq. rel. of free pmp actions of $\mathbb{F}_m$ and $\mathbb{F}_n$ are not orbit equivalent.

There is a converse to this corollary:

Theorem (Hjorth 2013, the lemma on cost achieved). If E is ergodic pmp treeable and of integer cost $n \in \mathbb{N} \cup \{\infty\}$, then E is induced by a free pmp action of $\mathbb{F}_n$.



There is an ergodic strengthening of this too:

Ergodic lemma on cost achieved (Miller–Ts) In Hjorth's theorem, the action of each of the n standard generators of $\mathbb{F}_n$ can be made ergodic.

Smooth equivalence relations. Recall that a Borel eq. rel. E on a st. Borel X is called **smooth** if $E \leq_B \mathrm{Id}_{\mathbb{R}}$, i.e. $\exists$ Borel function $\pi : X \to \mathbb{R}$ s.t. $\forall x_1, x_2 \in X$,

$$x_1 \, E \, x_2 \iff f(x_1) = f(x_2).$$

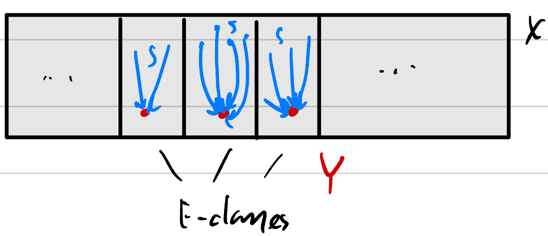
For CBERs stronger versions are available:

Prop. Let E be a CBER on X. TFAE:

(1) E is smooth.

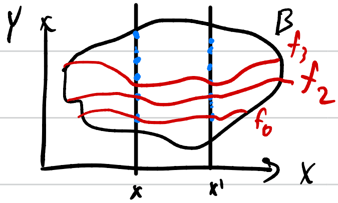
(2) E admits a **Borel selector**, i.e. a Borel map $s : X \to X$ s.t. $\forall x$, $s(x) \, E \, x$ and $\forall x_1, x_2 \in X$, $x_1 \, E \, x_2 \iff s(x_1) = s(x_2)$.

(3) E admits a Borel **transversal**, i.e. a Borel $Y \subseteq X$ that meets every E-class in exactly one point.



Proof. This follows from the Luzin–Novikov uniformization theorem. $\square$

Luzin–Novikov uniformization. Let $B \subseteq X \times Y$, X, Y st. Borel. If each X-fiber over B, namely, $B_x := \{ y \in Y : (x,y) \in B \}$, is ctbl, then $B = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n)$ for some Borel functions $f_n : X \to Y$.



In particular, $\mathrm{proj}_X B$ is Borel because $\mathrm{proj}_X B = \bigcup_{n \in \mathbb{N}} \mathrm{proj}\, \mathrm{graph}(f_n)$, and because proj on $\mathrm{graph}(f_n)$ is 1-1, its image is Borel (by the Luzin–Souslin).

Here is another characterization of smoothness that I find most useful.

<u>20-questions characterization of smoothness.</u> A CBER E on X is smooth iff $\exists\, (Q_n)_{n \in \mathbb{N}}$, $Q_n \subseteq X$ Borel ("questions") s.t. $\forall x_1, x_2$, $x_1 \mathrel{E} x_2 \iff \forall n\, (x_1 \in Q_n \iff x_2 \in Q_n)$.

Proof. $\Leftarrow$. Let (Q_n) be as in the hypothesis and define $\pi : X \to 2^{\mathbb{N}}$ by $x \mapsto$ sequence of answers, i.e. $(\mathbb{1}_{Q_n}(x))_{n \in \mathbb{N}}$. This is clearly a Borel reduction to $=$.

$\Rightarrow$. Let $\pi : X \to 2^{\mathbb{N}}$ be a Borel reduction to $=$. Then $Q_n := \pi^{-1}(\{x \in 2^{\mathbb{N}} : x(n) = 1\})$ fits the bill. $\square$

Examples. (a) Finite BERs (i.e. each class is finite) are smooth.

Proof. Suppose $X = \mathbb{R}$ and let $s : X \to X$ by $x \mapsto$ the least element in $[x]_E$. This is Borel by Lusin–Novikov.

(b) For any Borel function $f : X \to Y$, let $\ker(f) := \{(x_1, x_2) \in X^2 : f(x_1) = f(x_2)\}$, so it's smooth by def.

(c) Similarity of matrices.

(d) Conjugacy of Bernoulli automorphisms, by Ornstein's theorem.

Prop. Every ergodic CBER E on (X, μ) is μ-nowhere smooth, i.e. if $E|_Y$ is smooth then Y is null.

Proof. Ergodicity is eq. to every invariant meas. $\pi : X \to 2^{\mathbb{N}}$ is constant a.e. (recall taking preimages of left/right subtrees ). Since each E-class is ctbl, it's null, so every inv. meas. function would take inequivalent points to the same element in $2^{\mathbb{N}}$. $\square$

Examples. (a) E_0 on $2^{\mathbb{N}}$: $x \, E_0 \, y \iff \forall^\infty n \; x(n) = y(n)$.

This ergodic w.r.t. the coin-flip measure.

(b) Bernoulli shifts: $\forall$ ctbl grp Γ, take the shift $\Gamma \curvearrowright (X^\Gamma, \mu^\Gamma)$. This action is (strongly) mixing:

$$\lim_{\gamma \to \infty} \mu(\gamma \cdot A \cap B) = \mu(A)\mu(B),$$

where $\gamma \to \infty$ means $\forall \varepsilon > 0 \ \forall^\infty \gamma \in \Gamma \ldots$

Hence the orb. eq. rel. is ergodic, so nonsmooth.

(c) Irrational rotation $\mathbb{Z} \curvearrowright S^1$ is ergodic $\Rightarrow$ nonsmooth